Simplified Flow Calculations for Tubes and Parallel Plates

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The effective correlations of turbulent velocities and friction losses for tubes and parallel plates recently published have been analyzed further in order to simplify their use and to extend the range of Reynolds number.

Working diagrams have been developed from which turbulent friction losses and local velocities for tubes and parallel plates can be calculated without interpolation or trial-and-error procedures. Tentative values of parallel-plate friction factors and average-to-maximum velocity ratios in the transition region are also included, and new experimental values of the velocity ratio in smooth tubes are reported. The working diagrams permit more rapid, accurate, and consistent calculations of fluid behavior to be made over a wider range of operating conditions than was previously possible.

Successful use of the concept of hydraulic radius in calculating the friction head associated with turbulent flow in noncircular conduits hinges on the proper choice of an "equivalent" tube with which to compare the noncircular cross section. The most commonly used method is based on the assumption that a tube having the same ratio of cross-sectional area to wetted perimeter as the noncircular conduit is hydrodynamically equivalent to the noncircular configuration when operated at the same bulk average linear velocity with the same fluid.

Since there is much more reliable experimental information available about turbulent phenomena in tubes than in noncircular conduits, it is a practical necessity to base working correlations for the latter on a comparison of the two types of ducts. This fact does not, in itself, detract from utility and accuracy if a truly equivalent tube can be found. The results of recent experimental investigations of pressure drop and velocity distribution in annuli (*3, 9, 10*) have indicated, however, that the tube postulated in the usual hydraulic-radius concept is not the correct one, although its use often yields a good approximation to the actual friction head. It is therefore reasonable to believe that the hydraulic-radius method is based on an empiricism which in many cases is not far removed from the true situation.

In a recent article Rothfus and Monrad (*8*) presented correlations of pressure drop and velocity distribution in fluids flowing in turbulent motion between parallel flat plates. Their method involves the concept of an equivalent tube somewhat different from the one used in the method of hydraulic radius. The notable data of Sage and associates (*1, 6, 7*) show the new correlations to be effective for both pressure drop and velocity distribution. The equivalent tube used by Rothfus and Monrad is consistent with viscous-flow theory and appears to have some general significance in the handling of flow through noncircular ducts.

In order to use the new parallel-plate correlations for the prediction of friction losses and local velocities, it is necessary to perform a number of intermediate calculations which are troublesome and time consuming. This paper presents the Rothfus and Monrad correlations in a readily usable form, based on the best experimental data available, in which the intermediate calculations are implicit. To this end, some additional experimental values of the ratio of average-to-maximum velocity in smooth tubes are also presented.

BASIS FOR PARALLEL-PLATE CORRELATIONS

In truly viscous motion the velocity profile in a fluid of constant density flowing steadily and isothermally between two smooth, parallel, infinite, flat plates is a parabola. The same is true for similar flow in a smooth tube. Rothfus and Monrad have shown that the local velocities in the two types of conduits must be coincident if

1. The radius r_0 of the tube is equal to the half clearance b of the plates
2. The fluids have the same kinematic viscosity in each case
3. The friction velocity $\tau_0 g_0/\rho$ in the tube is equal to the friction velocity between the parallel plates.

The latter stipulation is equivalent to stating that the maximum (center-line) local velocities must be equal.

Rothfus and Monrad postulated that the same behavior might occur under comparable conditions in fully turbulent flow. On this basis they concluded that the friction velocity parameters as well as the friction distance parameters were the same in the two conduits at the same distance from the wall; that is,

$$(u^+)_p = (u^+)_F \qquad (1)$$

and

$$(y^+)_p = (y^+)_F \qquad (2)$$

whenever $r_0 = b$, $(\mu/\rho)_p = (\mu/\rho)_F$, and $(\sqrt{\tau_0 g_0/\rho})_p = (\sqrt{\tau_0 g_0/\rho})_F$. Also, in view of the restrictions placed on the densities and skin frictions, the Fanning friction factors were observed to be related through the equation

$$\frac{\sqrt{f_F}}{\sqrt{f_p}} = \frac{(V/u_m)_p}{(V/u_m)_F} \qquad (3)$$

Finally, the Reynolds numbers defined in the usual manner as $(N_{Re})_F = 4bV_F\rho/\mu$ and $(N_{Re})_p = 2r_0V_p\rho/\mu$ were found to be related through the equation

$$\frac{(N_{Re})_F}{(N_{Re})_p} = \frac{2(V/u_m)_F}{(V/u_m)_p} \qquad (4)$$

It is apparent that if the Reynolds number for the parallel-plate case under consideration is known, the Reynolds

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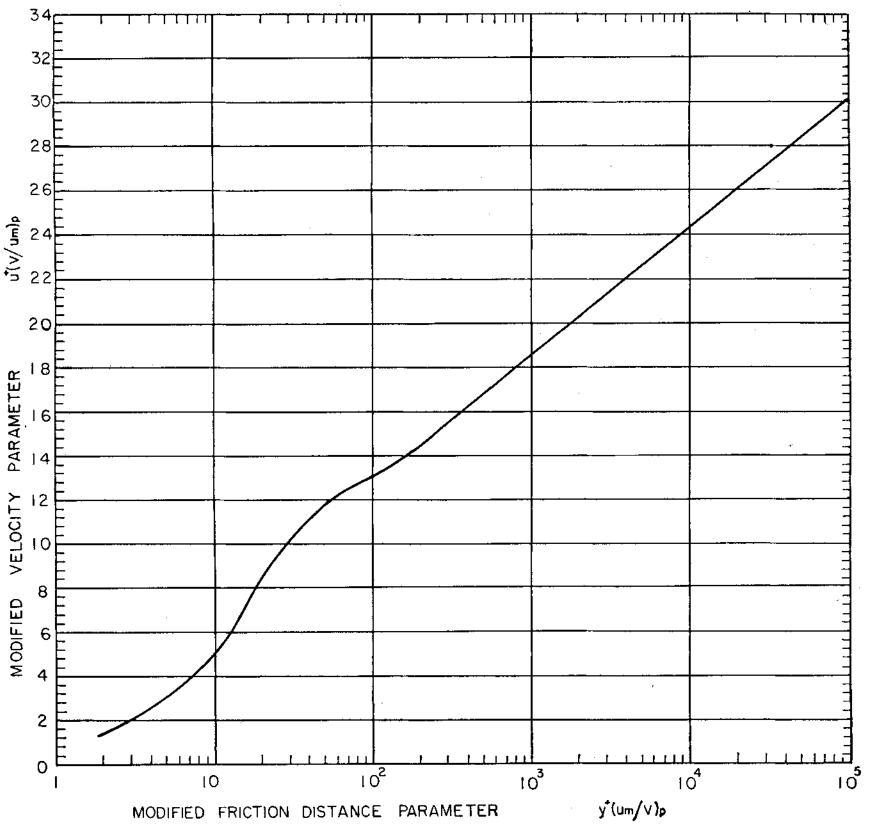

Fig. 1. Correlation of mean local velocities for steady, isothermal, turbulent flow in smooth tubes and between smooth, parallel, flat plates.

number at which the equivalent tube must be operated can be obtained from Equation (4) provided that the relationship between (V/u_m) and Reynolds number is available for each conduit. The friction factor for the tube is then obtained in the usual way at the tube Reynolds number and the friction factor for the parallel-plate case is calculated by means of Equation (3). The pressure drop

TABLE 1

Y^+	U^+				
	From	Deissler (tubes)		Sage (parallel plates)	
	Fig. 1	Min.	Max.	Min.	Max.
2	1.4	1.5	1.5		
4	2.6	2.3	2.9	2.0	2.5
6	3.4	3.3	4.0	3.1	3.7
10	5.1	4.9	5.6	4.7	5.9
20	8.4	8.1	9.1	8.2	8.9
40	11.1	10.4	11.1	10.1	11.0
60	12.1	11.3	12.3	11.6	12.2
100	13.1	12.5	13.4	12.7	13.4
200	14.4	13.8	14.8	14.0	14.5
400	16.2	16.2	17.1	15.4	16.2
600	17.2	16.9	17.5	16.8	17.3
1,000	18.7	18.5	19.1		
2,000	20.3	20.3	21.0		

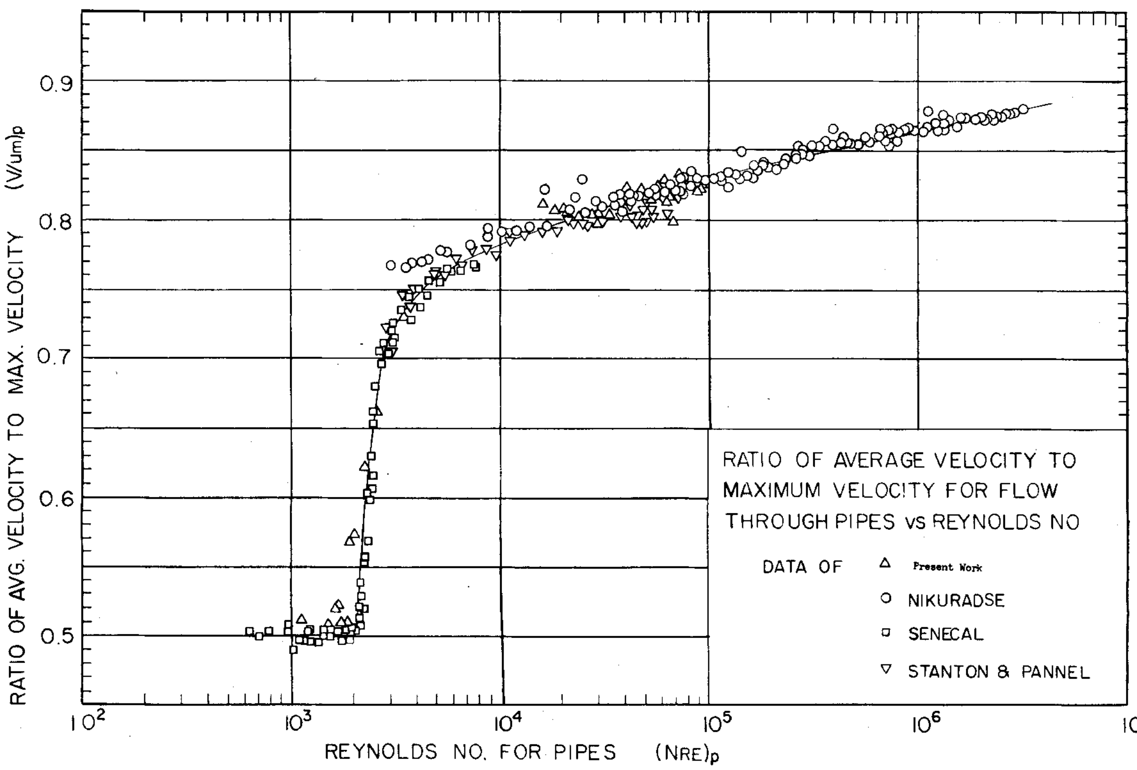

Fig. 2. Effect of Reynolds number on the ratio of bulk average to maximum velocity in smooth tubes.

due to friction can immediately be calculated from the Fanning equation written in terms of the clearance, $2b$, between the plates, namely

$$(\Delta p)_F = \frac{f_F \rho V_F^2 L}{g_0(2b)} \qquad (5)$$

If it is desired to obtain a value of the local velocity at some point in the stream, the friction velocity can be calculated by means of the equation

$$(u_*)_F = V_F \sqrt{f_F/2} \qquad (6)$$

and the values of $(y^+)_F$ and $(u^+)_F$ can be read directly from the u^+, y^+ correlation for smooth tubes, as indicated in Equations (1) and (2).

Unfortunately, two factors contribute to the difficulty of the calculations. In the first place, Equation (4) must be solved by means of trial and error if only the usual graph of $(V/u_m)_p$ against $(N_{Re})_p$ is available. In the second place, the u^+, y^+ relationship for smooth tubes is not unique but varies considerably with Reynolds number. This makes it necessary in some cases to perform a difficult interpolation.

The calculation of pressure-drop and velocity distribution can be simplified a great deal if the following diagrams are made available:

1. An accurate velocity correlation which does not require interpolation
2. Accurate values of the friction factor for parallel plates as a function of the Reynolds number for parallel plates.

In addition, it is sometimes convenient to have a working graph of the velocity ratio $(V/u_m)_F$ against the Reynolds number for parallel plates.

VELOCITY DISTRIBUTIONS FOR TUBES AND PARALLEL PLATES

Rothfus and Monrad have shown that the effect of Reynolds number can effectively be removed from the ordinary u^+, y^+ correlation for smooth tubes by empirical means. They have observed that a unique correlation can be obtained when the coordinates are modified to account for the change of (V/u_m) ratio with Reynolds number. Their modified coordinates are simply

$$U^+ = u^+ \left(\frac{V}{u_m}\right)_p = \frac{u}{u_*}\left(\frac{V}{u_m}\right)_p \qquad (7)$$

and

$$Y^+ = y^+ \left(\frac{u_m}{V}\right)_p = \frac{y u_* \rho}{\mu}\left(\frac{u_m}{V}\right)_p \qquad (8)$$

It follows from Equations (1) and (2) that the modified correlation must be equally valid for both tubes and parallel plates if the velocity profiles are actually coincident, as originally postulated. It should be noted that the $(V/u_m)_p$ term in the modified coordinates is that for the equivalent tube at the tube Reynolds number, since in view of Equations (1) and (2) the same correction must be made whether the tube case or parallel-plate case is being calculated.

As originally presented, the modified velocity correlation was based on the data of Nikuradse (*5*) and of Senecal and Rothfus (*11*) for the most part. Relatively few of the data were in the range of low Y^+ values (i.e., close to the tube wall), however; so there was considerable uncertainty attached to the portion of the experimental curve lying below a Y^+ of 30.

The extensive data of Deissler (*2*) have since been found to form a unique curve on U^+, Y^+ coordinates in the questionable range of Y^+ values. The resultant correlation, based on all the cited data for smooth tubes, is shown in Figure 1. The data of Sage and associates (*1, 6, 7*) on velocity distributions between parallel plates are in close agreement with Deissler's tube data as shown in Table 1.

Figure 1 is therefore equally applicable to both smooth tubes and smooth parallel plates and adequately represents the best available experimental data in each case. Its use is limited to the range of fully turbulent motion. On the basis of current information, it appears that the modified correlation is valid at Reynolds numbers greater than 3,000 for tubes and greater than 7,000 for parallel plates. It should be noted that the velocity curve cannot retain its unique character all the way to the tube wall, for viscous-flow theory predicts that U^+ must equal $[(V/u_m)_p^2 Y^+ (1 - y/2r_0)]$ in the laminar film adjacent to the solid surface.

The supporting data for tubes fully cover the Reynolds-number range from 3,000 to 3,240,000. Their average deviation from the curve shown in Figure 1 is essentially independent of the Reynolds number. The supporting data for parallel plates cover Reynolds numbers from 6,960 to 53,200. It appears reasonable, however, to predict that Figure 1 can be used for parallel plates up to Reynolds numbers of about ten million without appreciable error.

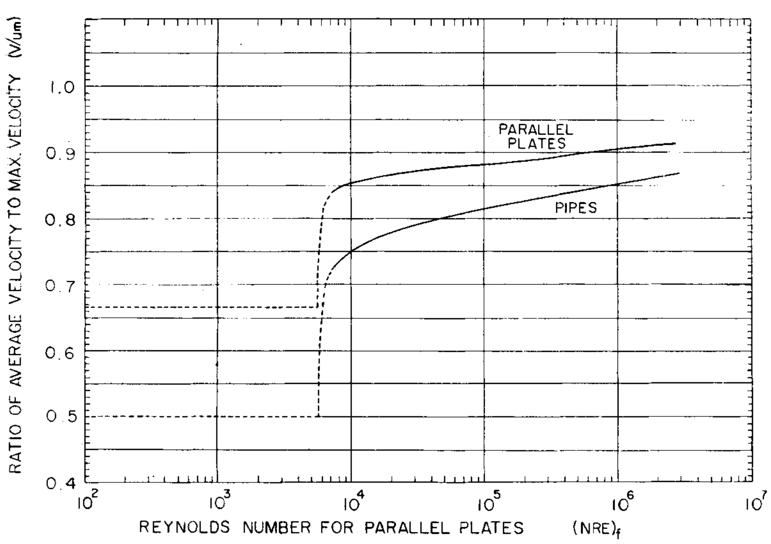

Fig. 3. Velocity ratios for parallel plates and "equivalent" smooth tubes as functions of the parallel-plate Reynolds number.

VELOCITY RATIOS IN TUBES

It is obvious that the modified velocity correlation requires accurate knowledge of the ratio of average to maximum velocity in smooth tubes over the entire turbulent range. The ratio is implicit in the correlation, of course, since integration of the velocity profile at any given Reynolds number must yield a consistent value of the bulk average velocity. The region near the tube wall, however, is a difficult one in which to obtain accurate measurements of the local velocity. Unfortunately, this region substantially influences the values of the bulk average velocity obtained by means of integration. Consequently, the average velocities obtained from experimental velocity profiles are often from 1 to 4% higher than those measured directly by means of flow-metering devices. In addition, there is an abundance of directly measured data in the literature on which a correlation of $(V/u_m)_p$ as a function of Reynolds number can be based.

Taken together, the data of Nikuradse (*5*), Stanton and Pannell (*12*), and Senecal and Rothfus (*11*) furnish experimental values of $(V/u_m)_p$ over the entire range of Reynolds numbers from 600 to 3,240,000. The viscous, transition, and turbulent regimes are therefore included and a complete picture of the effect of Reynolds number is available. There is only one portion of the flow range in which the data are inconsistent. At Reynolds numbers between 4,000 and 100,000 Nikuradse's values are significantly higher than those of Stanton and Pannell. Inconsistencies in Nikuradse's local velocity data in the lower turbulent region lend the impression that the Stanton and Pannell data are more reliable below a Reynolds number of, say, 50,000. This view is supported by the fact that the latter data agree with Senecal's data in the low turbulent region where the two overlap. On the other hand, Stanton's data terminate at a Reynolds number of about 100,000 and cannot be extrapolated to agreement with Nikuradse's high Reynolds-number values.

In order to reconcile the differences, experimental values of $(V/u_m)_p$ have been obtained at Reynolds numbers between 20,000 and 100,000 in a brass tube having an inside diameter of 1.00 in. The test fluid was water at room temperature. The bulk average velocity was obtained by means of a calibrated orifice meter. The maximum local velocity was measured by means of an impact tube made from hypodermic tubing having an outside diameter of 0.042 in. It was found that the data thus obtained formed a satisfactory connection between the low Reynolds-number data of Stanton and Pannell and the high Reynolds-number data of Nikuradse.

Figure 2 summarizes the effect of Reynolds number on the ratio of average to maximum velocity in smooth tubes. The solid line indicates the recommended values of $(V/u_m)_p$ to be used in conjunction with the velocity correlation shown in Figure 1. In order to present the complete correlation on the same set of coordinates, the recommended line has been extended through the transition region. It should be remembered, however, that Figure 1 is valid only at Reynolds numbers greater than 3,000.

VELOCITY RATIOS BETWEEN PARALLEL PLATES

In order to apply the velocity correlation shown in Figure 1 to flow between smooth parallel plates, it is necessary to determine the Reynolds number in the equivalent tube. The reason for this stems directly from the fact that $(V/u_m)_p$ must be evaluated at the tube Reynolds number. It would therefore prove convenient to have available a diagram showing $(V/u_m)_p$ as a function of the parallel-plate Reynolds number since the latter is usually a known starting point for velocity computations.

Since the velocity profiles for the tube and parallel plates are coincident at the conditions under which the comparison is made, it is possible to develop the relationship between the tube and parallel-plate Reynolds numbers in the following manner.

By definition,

$$V_p = 2 \int_0^1 u_p \left(1 - \frac{y}{r_0}\right) d\left(\frac{y}{r_0}\right) \quad (9)$$

and

$$V_F = \int_0^1 u_F \, d\left(\frac{y}{b}\right) = \int_0^1 u_F \, d\left(\frac{y}{r_0}\right) \quad (10)$$

Therefore

$$(N_{Re})_p = \frac{4r_0\rho}{\mu} \int_0^1 u_p \, d\left(\frac{y}{r_0}\right) - \frac{4r_0\rho}{\mu} \int_0^1 u_p \left(\frac{y}{r_0}\right) d\left(\frac{y}{r_0}\right) \quad (11)$$

and similarly,

$$(N_{Re})_F = \frac{4r_0\rho}{\mu} \int_0^1 u_F \, d\left(\frac{y}{r_0}\right) \quad (12)$$

But the velocity profiles are coincident, and so the local velocity u_p is equal to the local velocity u_F. Consequently

$$(N_{Re})_F = (N_{Re})_p + \frac{4r_0\rho}{\mu} \int_0^1 u_p \left(\frac{y}{r_0}\right) d\left(\frac{y}{r_0}\right) \quad (13)$$

The last equation can be combined with Equations (7) and (8) to yield an expression for the Reynolds-number relationship in terms of the modified friction velocity parameter and friction distance parameter, namely

$$(N_{Re})_F = (N_{Re})_p + \frac{4}{Y_m^+} \int_0^{Y_m^+} U^+ \, Y^+ \, dY^+ \quad (14)$$

where Y_m^+ is the maximum (or centerline) value of the modified friction distance parameter.

The integration indicated in Equation (14) has been performed by means of Simpson's rule. In the fully turbulent range corresponding values of U^+ and Y^+ were taken from Figure 1. The velocity ratio $(V/u_m)_p$ was obtained from Figure 2. Fanning friction factors f_p were evaluated by means of Koo's equation for smooth tubes (*4*).

Almost nothing is known about the transition behavior of fluids flowing between parallel plates. On the other hand, the velocity distribution and friction data of Senecal and Rothfus cover the transition range for smooth tubes. Since the method of comparing tubes and parallel plates suggested by Rothfus and Monrad appears to be equally valid for viscous and fully turbulent flow, it might be reasonable to postulate its validity in the transition range as well.

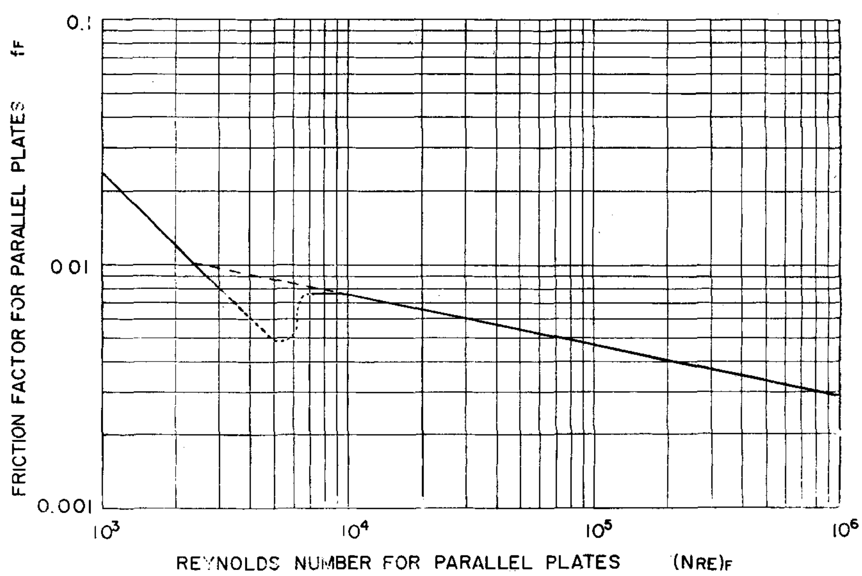

Fig. 4. Effect of Reynolds number on the Fanning friction factor for parallel plates.

It is recognized that such an assumption must be considered highly tentative, subject to future experimental investigation. On this basis the velocity data of Senecal and Rothfus have been integrated as indicated in Equation (14). Friction factors were obtained from their friction data in the transition range and $(V/u_m)_p$ values were again taken from Figure 2.

The resultant diagram of $(V/u_m)_p$ against the parallel-plate Reynolds number is shown in Figure 3. The transition relationship is shown as a dashed line to emphasize its tentative nature. In the fully turbulent region, the values of $(V/u_m)_p$ to be used in conjunction with Figure 1 can be read directly at the parallel-plate Reynolds number.

It is sometimes convenient to have a diagram of $(V/u_m)_F$ against $(N_{Re})_F$ immediately available. Such a curve is also included in Figure 3. It was obtained through combination of Equations 4 and 14 into the expression

$$\left(\frac{V}{u_m}\right)_F = \frac{1}{2}\left(\frac{V}{u_m}\right)_p + \frac{\sqrt{f_p/2}}{(Y_m^+)^2} \cdot \int_0^{Y_m^+} U^+ \, Y^+ \, dY^+ \qquad (15)$$

from which $(V/u_m)_F$ was calculated at various tube Reynolds numbers. The transition range has been treated in the manner previously described, and the resultant relationship is shown as the dashed portion of the curve.

Both curves in Figure 3 are made consistent with theory in the viscous-flow range. Senecal's data indicate that the ratio of average to maximum velocity in smooth tubes deviates negligibly from the viscous-range value of 0.500 at Reynolds numbers below 2,100. Since the theoretical value of $(V/u_m)_F$ is 0.667 in the viscous range, Equation (4) suggests that this value should be maintained up to a Reynolds number of 5,600 in the parallel-plate case.

FRICTION FACTORS FOR PARALLEL PLATES

The Fanning friction factors for parallel plates have been calculated by means of Equation (3) and are presented as functions of the parallel-plate Reynolds number in Figure 4. The indicated values are based on velocity ratios from Figure 3, Koo's friction factor equation in the turbulent range, and Senecal's friction data in the transition range. The theoretical value of $24/(N_{Re})_F$ is shown in the viscous region. The transition range is again indicated by a dashed line since the validity of Equation (3) in this region has been assumed without supporting data. A few experimental points have been obtained in rectangular ducts (*13*) and these lie within the triangular area bounded by the transition curve and the extension of the turbulent line. The transition region appears to extend upward to a Reynolds number of about 7,000. The same upper limit has been found to exist at the outer walls of annuli by Rothfus, Monrad, Sikchi, and Heideger (*10*).

USE OF THE CORRELATIONS

Figures 1 through 4 afford accurate and consistent means by which to calculate friction losses and local velocities associated with isothermal turbulent flow in smooth tubes and between smooth parallel plates. The correlations can be used directly without any necessity for intermediate calculations involving trial-and-error procedures.

If it is desired to calculate the pressure drop caused by fluid friction between parallel plates, the friction factor obtained from Figure 4 at the parallel-plate Reynolds number can be inserted directly into Equation (5). If the local velocity at some point between the plates is desired, the proper value of $(V/u_m)_p$ can be read from Figure 3 at the parallel-plate Reynolds number. Figure 1 can then be used in conjunction with the friction factor from Figure 4 to obtain the desired velocity. If a local velocity within a smooth tube is to be calculated, $(V/u_m)_p$ can be obtained from Figure 2 and the velocity can be determined by means of Figure 1 and appropriate friction factors.

Although tentative values of the friction factor and velocity ratio for parallel plates are presented in the transition range, such values must be used with caution until experimental data are made available.

Since $(V/u_m)_p$ varies only from 0.80 at a Reynolds number of 25,000 to 0.88 at a Reynolds number of 3,000,000, it is apparent that the Reynolds-number effect on the unmodified u^+, y^+ diagram is very small in the higher turbulent range. The modified correlation is therefore most useful at Reynolds numbers between 3,000 and 25,000 where the main-stream velocity profile is appreciably affected by the Reynolds number. If it is desired to obtain local velocities close to the wall at high Reynolds numbers, however, the modified correlation affords a reasonable means of extrapolating main-stream data into the wall region where no reliable experimental information is available at such Reynolds numbers.

NOTATION

b = half clearance between parallel plates, ft.
f Fanning friction factor $[=(2R_H\Delta p g_0)/(\rho V^2 L)]$, dimensionless
g_0 = conversion factor = 32.2 (lb.mass)(ft.)/(lb.force)(sec.)(sec.)
L = length of conduit over which Δp is measured, ft.
N_{Re} = Reynolds number $(= 4R_H V\rho/\mu)$, dimensionless
Δp = pressure drop caused by fluid friction, lb. force/sq. ft.
r_0 = radius of experimental or "equivalent" tube, ft.
R_H = hydraulic radius ($= r_0/2$ for tubes and $= b$ for parallel plates), ft.
u = local fluid velocity, ft./sec.
u_m = maximum local fluid velocity, ft./sec.
u_* = friction velocity $(= \sqrt{\tau_0 g_0/\rho} = V\sqrt{f/2})$, ft./sec.
u^+ = friction velocity parameter $(= u/u_*)$, dimensionless
U^+ = modified friction velocity parameter defined by Equation (7), dimensionless
V = bulk average linear velocity, ft./sec.
y = distance from the conduit wall at which the local velocity u is measured, ft.
y^+ = friction distance parameter $(= yu_*\rho/\mu)$, dimensionless
Y^+ = modified friction distance parameter defined by Equation (8), dimensionless
Y_m^+ = maximum value of the friction distance parameter, dimensionless
μ = fluid viscosity, lb. mass/(sec.)(ft.)
ρ = fluid density, lb. mass/cu. ft.
τ_0 = skin friction at the conduit wall, lb. force/sq. ft.

Subscripts

F = flow between parallel plates
p = flow in tubes

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